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REDUCTION OF THE PROBLEM OF THREE BODIES.

BY G. W. HILL, NYACK TURNPIKE, N. Y.

THE object of this article is to find the three differential equations which virtually determine the sides of the triangle formed by the three bodies, bringing to ones aid all the known finite integrals of the problem.

Lagrange was the first to treat this question in his *Essai sur le Problème des Trois Corps* (*Oeuvres*, Tome VI, p. 227); but the formulæ lacking symmetry, his editor, Serret, has, in a note, supplied this and pointed out an important error into which Otto Hesse, who had investigated this subject, (Journal für die Mathematik, Band LXXIV) had fallen.

By adopting an orthogonal substitution, at the outset, for reducing the number of coordinates from nine to six, we can prevent the masses from entering the equations except through the potential function or its derivatives. In this way symmetry indeed appears to be lost, but there is so great a gain in condensation of the formulæ, that we can carry out some of the eliminations which the previous writers have been content only to indicate.

Let ξ, η, ζ; ξ', η', ζ'; ξ'', η'', ζ'' be the rectangular coordinates of the masses m, m', m'', the expression for the living force will be

$$T = m\frac{d\xi^2+d\eta^2+d\zeta^2}{2dt^2} + m'\frac{d\xi'^2+d\eta'^2+d\zeta'^2}{2dt^2} + m''\frac{d\xi''^2+d\eta''^2+d\zeta''^2}{2dt^2};$$

and Δ, Δ', Δ'' being given by the equations

$$\begin{aligned} \Delta^2 &= (\xi' - \xi'')^2 + (\eta' - \eta'')^2 + (\zeta' - \zeta'')^2, \\ \Delta'^2 &= (\xi'' - \xi\ )^2 + (\eta'' - \eta\ )^2 + (\zeta'' - \zeta\ )^2, \\ \Delta''^2 &= (\xi\ - \xi'\ )^2 + (\eta\ - \eta'\ )^2 + (\zeta\ - \zeta'\ )^2, \end{aligned}$$

the potential function

$$\Omega = \frac{m'm''}{\Delta} + \frac{mm''}{\Delta'} + \frac{mm'}{\Delta''}.$$

Without lessening the generality, the origin of coordinates can be put at the centre of gravity, when the principle of the conservation of this centre will furnish the equations

$$\left.\begin{aligned} m\xi + m'\xi' + m''\xi'' &= 0, \\ m\eta + m'\eta' + m''\eta'' &= 0, \\ m\zeta + m'\zeta' + m''\zeta'' &= 0. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (1)$$

By means of these relations three of the variables can be eliminated and the number thus reduced from 9 to 6. This transformation is most elegantly accomplished by putting

$$\begin{aligned}
\xi &= \alpha x + \beta x', & \eta &= \alpha y + \beta y', & \zeta &= \alpha z + \beta z',\\
\xi' &= \alpha' x + \beta' x', & \eta' &= \alpha' y + \beta' y', & \zeta' &= \alpha' z + \beta' z',\\
\xi'' &= \alpha'' x + \beta'' x', & \eta'' &= \alpha'' y + \beta'' y', & \zeta'' &= \alpha'' z + \beta'' z',
\end{aligned}$$

where α, α', α'', β, β', β'' are six constants which may be so taken that they satisfy the five equations

$$\left.\begin{aligned}
m\alpha + m'\alpha' + m''\alpha'' &= 0,\\
m\beta + m'\beta' + m''\beta'' &= 0,\\
m\alpha\beta + m'\alpha'\beta' + m''\alpha''\beta'' &= 0,\\
m\alpha^2 + m'\alpha'^2 + m''\alpha''^2 &= 1,\\
m\beta^2 + m'\beta'^2 + m''\beta''^2 &= 1.
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (2)$$

The first two are necessary in order that the equations (1) may be satisfied; the third is adopted in order that nothing but squares of differential coefficients may occur in the transformed T; and, evidently, the last two may be adopted without thereby diminishing the generality of the transformation.

These equations may be solved elegantly in the following manner. Put

$$\sqrt{m} = k \sin\gamma \cos\varepsilon, \qquad \sqrt{m'} = k \sin\gamma \sin\varepsilon, \qquad \sqrt{m''} = k \cos\gamma;$$

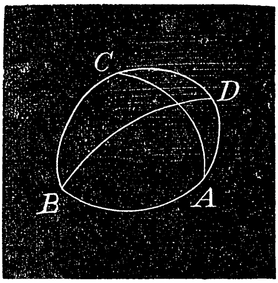
and adopt the four quantities φ, φ', ω, ω' such that

$$\begin{aligned}
\sqrt{m}\,\alpha &= \sin\varphi \cos(\omega+\varepsilon), & \sqrt{m}\,\beta &= \sin\varphi' \cos(\omega'+\varepsilon),\\
\sqrt{m'}\,\alpha' &= \sin\varphi \sin(\omega+\varepsilon), & \sqrt{m'}\,\beta' &= \sin\varphi' \sin(\omega'+\varepsilon),\\
\sqrt{m''}\,\alpha'' &= \cos\varphi, & \sqrt{m''}\,\beta'' &= \cos\varphi';
\end{aligned}$$

it is plain that the last two of equations (2) will be satisfied, and the first three take the forms

$$\begin{aligned}
\cos\gamma\cos\varphi + \sin\gamma\sin\varphi\cos\omega &= 0,\\
\cos\gamma\cos\varphi' + \sin\gamma\sin\varphi'\cos\omega' &= 0,\\
\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi'\cos(\omega-\omega') &= 0.
\end{aligned}$$

Hence if the quadrantal spherical triangle ABC is constructed, and the arc $AD = \gamma$, having any arbitrary orientation on the sphere, drawn, and BD and CD joined, we shall have $\varphi = BD$, $\varphi' = CD$, $\omega = ADB$, $\omega' = ADC$ as the general solution of the system of equations (2).



Then, after substitution,

$$T = \frac{dx^2+dy^2+dz^2}{2dt^2} + \frac{dx'^2+dy'^2+dz'^2}{2dt^2}.$$

And if we put

$$\begin{aligned}
f &= \alpha' - \alpha'', & f' &= \alpha'' - \alpha, & f'' &= \alpha - \alpha',\\
g &= \beta' - \beta'', & g' &= \beta'' - \beta, & g'' &= \beta - \beta',\\
v &= x^2 + y^2 + z^2, & v' &= x'^2 + y'^2 + z'^2, & v'' &= xx' + yy' + zz',
\end{aligned}$$

we have the expressions

$$\Delta^2 = f^2 v + 2fgv'' + g^2 v',$$
$$\Delta'^2 = f'^2 v + 2f'g'v'' + g'^2 v',$$
$$\Delta''^2 = f''^2 v + 2f''g''v'' + g''^2 v'.$$

The equations of motion are now

$$\frac{d^2x}{dt^2} = \frac{d\Omega}{dx}, \qquad \frac{d^2y}{dt^2} = \frac{d\Omega}{dy}, \qquad \frac{d^2z}{dt^2} = \frac{d\Omega}{dz},$$
$$\frac{d^2x'}{dt^2} = \frac{d\Omega}{dx'}, \qquad \frac{d^2y'}{dt^2} = \frac{d\Omega}{dy'}, \qquad \frac{d^2z'}{dt^2} = \frac{d\Omega}{dz'}.$$

Or, regarding Ω as a function of v, v', v'',

$$\frac{d^2x}{dt^2} = 2x\frac{d\Omega}{dv} + x'\frac{d\Omega}{dv''}, \qquad \frac{d^2x'}{dt^2} = 2x'\frac{d\Omega}{dv'} + x\frac{d\Omega}{dv''},$$
$$\frac{d^2y}{dt^2} = 2y\frac{d\Omega}{dv} + y'\frac{d\Omega}{dv''}, \qquad \frac{d^2y'}{dt^2} = 2y'\frac{d\Omega}{dv'} + y\frac{d\Omega}{dv''},$$
$$\frac{d^2z}{dt^2} = 2z\frac{d\Omega}{dv} + z'\frac{d\Omega}{dv''}, \qquad \frac{d^2z'}{dt^2} = 2z'\frac{d\Omega}{dv'} + z\frac{d\Omega}{dv''}.$$

From these, by eliminating the partial derivatives of Ω, we obtain

$$\frac{xd^2y - yd^2x}{dt^2} + \frac{x'd^2y' - y'd^2x'}{dt^2} = 0,$$
$$\frac{zd^2x - xd^2z}{dt^2} + \frac{z'd^2x' - x'd^2z'}{dt^2} = 0,$$
$$\frac{yd^2z - zd^2y}{dt^2} + \frac{y'd^2z' - z'd^2y'}{dt^2} = 0.$$

The integrals of which are

$$\frac{xdy - ydx}{dt} + \frac{x'dy' - y'dx'}{dt} = k\cos\mu,$$
$$\frac{zdx - xdz}{dt} + \frac{z'dx' - x'dz'}{dt} = k\sin\mu\cos\nu,$$
$$\frac{ydz - zdy}{dt} + \frac{y'dz' - z'dy'}{dt} = k\sin\mu\sin\nu,$$

k, μ and ν being the arbitrary constants. In addition there is the integral of living forces

$$T = \Omega + h.$$

If we put

$$u = \frac{dx^2 + dy^2 + dz^2}{dt^2}, \quad u' = \frac{dx'^2 + dy'^2 + dz'^2}{dt^2}, \quad u'' = \frac{dxdx' + dydy + dzdz'}{dt^2},$$

it is evident that

$$\frac{1}{2}\frac{d^2v}{dt^2} - u = \frac{xd^2x + yd^2y + zd^2z}{dt^2},$$

$$\frac{1}{2}\frac{d^2v'}{dt^2} - u' = \frac{x'd^2x' + y'd^2y' + z'd^2z'}{dt^2},$$

$$\frac{d^2v''}{dt^2} - u'' = \frac{xd^2x' + yd^2y' + zd^2z' + x'd^2x + y'd^2y + z'd^2z}{dt^2}.$$

We put moreover

$$\rho = \frac{xdx' - x'dx + ydy' - y'dy + zdz' - z'dz}{2dt},$$

whence

$$\frac{xdx' + ydy' + zdz'}{dt} = \frac{1}{2}\frac{dv''}{dt} + \rho,$$

$$\frac{x'dx + y'dy + z'dz}{dt} = \frac{1}{2}\frac{dv''}{dt} - \rho.$$

We have

$$\frac{d\rho}{dt} = \frac{xd^2x' - v'd^2x + yd^2y' - y'd^2y + zd^2z' - z'd^2z}{2dt^2}.$$

By the substitution of the values of $\frac{d^2x}{dt^2}$, &c., we obtain

$$\frac{1}{2}\frac{d^2v}{dt^2} - u = 2v\frac{d\Omega}{dv} + v''\frac{d\Omega}{dv''},$$

$$\frac{1}{2}\frac{d^2v'}{dt^2} - u' = 2v'\frac{d\Omega}{dv'} + v''\frac{d\Omega}{dv''},$$

$$\frac{d^2v''}{dt^2} - u'' = 2v''\left(\frac{d\Omega}{dv} + \frac{d\Omega}{dv'}\right) + (v + v')\frac{d\Omega}{dv''},$$

$$\frac{d\rho}{dt} = v''\left(\frac{d\Omega}{dv'} - \frac{d\Omega}{dv}\right) + \tfrac{1}{2}(v - v')\frac{d\Omega}{dv''}.$$

These equations take simpler forms when the variables are changed as follows,

$$\mathrm{w} = \tfrac{1}{4}(v + v'), \qquad \mathrm{w}' = \tfrac{1}{4}(v - v'), \qquad \mathrm{w}'' = \tfrac{1}{2}v'',$$

$$\nu = \tfrac{1}{2}(u + u'), \qquad \nu' = \tfrac{1}{2}(u - u'), \qquad \nu'' = u''.$$

Then they become

$$\left.\begin{aligned}
\frac{d^2\mathrm{w}}{dt^2} - \nu &= \mathrm{w}\frac{d\Omega}{d\mathrm{w}} + \mathrm{w}'\frac{d\Omega}{d\mathrm{w}'} + \mathrm{w}''\frac{d\Omega}{d\mathrm{w}''} = -\tfrac{1}{2}\Omega,\\
\frac{d^2\mathrm{w}'}{dt^2} - \nu' &= \mathrm{w}'\frac{d\Omega}{d\mathrm{w}} + \mathrm{w}\frac{d\Omega}{d\mathrm{w}'},\\
\frac{d^2\mathrm{w}''}{dt^2} - \tfrac{1}{2}\nu'' &= \mathrm{w}''\frac{d\Omega}{d\mathrm{w}} + \mathrm{w}\frac{d\Omega}{d\mathrm{w}''},\\
\frac{d\rho}{dt} &= \mathrm{w}'\frac{d\Omega}{d\mathrm{w}''} - \mathrm{w}''\frac{d\Omega}{d\mathrm{w}'}.
\end{aligned}\right\} \quad \ldots\ldots (3)$$

If we add to the first of these the equation of living forces $\nu = \Omega + h$, we get

$$\frac{d^2\mathrm{w}}{dt^2} = \tfrac{1}{2}\Omega + h, \dots\dots\dots\dots\dots \quad (4)$$

an equation involving only the variables w, w′ and w″.

If we square the members of the three equations which constitute the principle of conservation of areas, and take their sum, the result will evidently be an equation which is not changed by a cyclical permutation f the letters x, y, z.

We have the identical equation

$$(xdy - ydx)^2 + (zdx - xdz)^2 + (ydz - zdy)^2 = (x^2+y^2+z^2)(dx^2+dy^2+dz^2) \\ - (xdx+ydy+zdz)^2,$$

with the similar equation which is obtained by affixing accents to x, y, z.

In addition there is the identity

$$(xdy-ydx)(x'dy'-y'dx')+(zdx-xdz)(z'dx'-x'dz')+(ydz-zdy)(y'dz'-z'dy')= \\ (xx'+yy'+zz')(dxdx'+dydy'+dzdz')-(xdx'+ydy'+zdz')(x'dx+y'dy+z'dz).$$

From these equations it will be seen that the equation of the sum of the squares takes the form

$$vu+v'u'+2v''u'' - \frac{dv^2+dv'^2+dv''^2}{4dt^2} + 2\rho^2 = k^2,$$

or, after transforming into terms of the new variables, and for convenience writing k for $\tfrac{1}{2}k$

$$\rho^2 = 2k^2 + \frac{d\mathrm{w}^2 + d\mathrm{w}'^2 + d\mathrm{w}''^2}{dt^2} - 2(\mathrm{w}\nu + \mathrm{w}'\nu' + \mathrm{w}''\nu'', \dots \quad (5)$$

an equation which is symmetrical.

It is evident now that, since the values of ν, ν' and ν'' are known from the first three equations of (3), we shall have, as the equations determining w, w′ and w″, (4), (5) and the last of (3), provided we can find a relation connecting ρ with w, w′, w″, ν, ν', ν'' and the differentials of the first three.

Such a relation can be found in the following manner:—assume the four indeterminates X, X', X'', X''' so that the equations

$$xX + x'X' + \frac{dx}{dt}X'' + \frac{dx'}{dt}X''' = 0,$$

$$yX + y'X' + \frac{dy}{dt}X'' + \frac{dy'}{dt}X''' = 0,$$

$$zX + z'X' + \frac{dz}{dt}X'' + \frac{dz'}{dt}X''' = 0,$$

are satisfied; and treat the last as if they were equations of condition in the method of least squares, that is, multiply the first by x, the second by y, the third by z, and take the sum for a first equation; and so on. In this way the normal equations formed from them are

$$vX + v''X' + \frac{1}{2}\frac{dv}{dt}X'' + \left(\frac{1}{2}\frac{dv''}{dt} + \rho\right)X''' = 0,$$

$$v''X + v'X' + \left(\frac{1}{2}\frac{dv''}{dt} - \rho\right)X'' + \frac{1}{2}\frac{dv'}{dt}X''' = 0,$$

$$\frac{1}{2}\frac{dv}{dt}X + \left(\frac{1}{2}\frac{dv''}{dt} - \rho\right)X' + u\,X'' + u''X''' = 0,$$

$$\left(\frac{1}{2}\frac{dv''}{dt} + \rho\right)X + \frac{1}{2}\frac{dv'}{dt}X' + u''X'' + u'X''' = 0.$$

As the number of these equations exceeds that of those from which they were derived, they are not independent, and the determinant, formed from the coefficients, vanishes, which is the condition determining ρ. This equation is

$$\begin{aligned}
&\left[\rho^2 + \frac{dvdv' - dv''}{4dt^2}\right]^2 + (vv' - v''^2)(uu' - u''^2)\\
&-\left[v'\rho^2 + \frac{v'dv'' - v''dv}{dt}\rho + \frac{v'dv''^2 - 2v''dv'dv'' + vdv'^2}{4dt^2}\right]u\\
&-\left[v\,\rho^2 + \frac{v''dv - vdv''}{dt}\rho + \frac{vdv''^2 - 2v''dvdv'' + v'dv^2}{4dt^2}\right]u'\\
&+2\left[v''\rho^2 + \frac{v'dv - vdv'}{2dt}\rho + \frac{(v'dv + vdv')dv'' - v''dvdv' - v''dv''^2}{4dt^2}\right]u'' = 0,
\end{aligned}$$

or, expressed in terms of the new variables,

$$\begin{aligned}
&\left[\rho^2 + \frac{d\mathrm{w}^2 - d\mathrm{w}'^2 - d\mathrm{w}''^2}{dt^2}\right]^2 + 4[\mathrm{w}^2 - \mathrm{w}'^2 - \mathrm{w}''^2]\,[\nu^2 - \nu'^2 - \nu''^2]\\
&-4\,[\mathrm{w}\nu - \mathrm{w}'\nu' - \mathrm{w}''\nu'']\rho^2\\
&+8\left[\frac{\mathrm{w}'d\mathrm{w}'' - \mathrm{w}''d\mathrm{w}'}{dt}\nu + \frac{\mathrm{w}''d\mathrm{w} - \mathrm{w}d\mathrm{w}''}{dt}\nu' + \frac{\mathrm{w}d\mathrm{w}' - \mathrm{w}'d\mathrm{w}}{dt}\nu''\right]\rho\\
&-4\left[\mathrm{w}\,\frac{d\mathrm{w}^2 + d\mathrm{w}'^2 + d\mathrm{w}''^2}{dt^2} - \frac{d\mathrm{w}}{dt}\frac{d(\mathrm{w}'^2 + \mathrm{w}'^2)}{dt}\right]\nu\\
&-4\left[\mathrm{w}'\frac{d\mathrm{w}^2 + d\mathrm{w}'^2 - d\mathrm{w}''^2}{dt^2} - \frac{d\mathrm{w}'}{dt}\frac{d(\mathrm{w}^2 - \mathrm{w}''^2)}{dt}\right]\nu'\\
&-4\left[\mathrm{w}''\frac{d\mathrm{w}^2 - d\mathrm{w}'^2 + d\mathrm{w}''^2}{dt^2} - \frac{d\mathrm{w}''}{dt}\frac{d(\mathrm{w}^2 - \mathrm{w}'^2)}{dt}\right]\nu'' = 0.
\end{aligned}$$

If ρ is eliminated from this equation by means of its value from (5), we shall have an equation of the first degree in ρ, from which the value of this quantity can be derived; this value will be written below.

In *resumé*, we can present our results as follows:— Let the five symbols, Ω, υ,* υ', υ'' and ρ, have the signifiications

$$\Omega = [aw + a'w' + a''w'']^{-\frac{1}{2}} + [bw + b'w' + b''w'']^{-\frac{1}{2}} + [cw + c'w' + c''w'']^{-\frac{1}{2}},$$

where a, b, c, &c., denote certain functions of the masses and of a single arbitrary quantity,

$$\upsilon = \frac{d^2w}{dt^2} - w\frac{d\Omega}{dw} - w'\frac{d\Omega}{dw'} - w''\frac{d\Omega}{dw''} = \frac{d^2w}{dt^2} + \tfrac{1}{2}\Omega,$$

$$\upsilon' = \frac{d^2w'}{dt^2} - w'\frac{d\Omega}{dw} - w\frac{d\Omega}{dw'},$$

$$\tfrac{1}{2}\upsilon'' = \frac{d^2w''}{dt^2} - w''\frac{d\Omega}{dw} - w\frac{d\Omega}{dw''},$$

$$\rho = \frac{\left\{\begin{array}{l} \left[\frac{dw^2}{dt^2} - 2w\upsilon + k^2\right]^2 \\ -\ [(w\upsilon' - w'\upsilon)^2 + (w''\upsilon - w\upsilon'')^2 - (w''\upsilon' - w'\upsilon'')^2] \\ -2\left[\frac{dw'}{dt}\,\frac{wdw' - w'dw}{dt} + \frac{dw''}{dt}\,\frac{wdw'' - w''dw}{dt}\right]\upsilon \\ -2\left[\frac{dw}{dt}\,\frac{w'dw - wdw'}{dt} + \frac{dw''}{dt}\,\frac{w''dw' - w'dw''}{dt}\right]\upsilon' \\ -2\left[\frac{dw}{dt}\,\frac{w''dw - wdw''}{dt} + \frac{dw'}{dt}\,\frac{w'dw'' - w''dw'}{dt}\right]\upsilon'' \end{array}\right\}}{2\left[\frac{w''dw' - w'dw''}{dt}\upsilon + \frac{wdw'' - w''dw}{dt}\upsilon' + \frac{w'dw - wdw'}{dt}\upsilon''\right]}$$

Then the differential equntions, which determine w, w′ and w″, are

$$\frac{d^2w}{dt^2} = \tfrac{1}{2}\Omega + h,$$

$$\rho^2 = 2k^2 + \frac{dw^2 + dw'^2 + dw''^2}{dt^2} - 2(w\upsilon + w'\upsilon' + w''\upsilon''),$$

$$\frac{d\rho}{dt} = w'\frac{d\Omega}{dw''} - w''\frac{d\Omega}{dw'}.$$

The first and second are of the second order, while the third is of the third order. It will be noticed that although the expressions, involved in them, are not exactly symmetrical, yet they exhibit some approach to symmetry; and perhaps by a slight change in the variables they may be made so. But I have not succeeded in discovering such a transformation.

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*For want of sorts the Greek *u* is here, and throughout the remainder of this article, written for *v*.—*Compositor.*